Unraveling Euler's Number: Historical Perspectives and Modern Calculation Methods

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Overview

Introduction

- Euler's number (e) is fundamental to mathematics.
- Applications range from continuous growth to complex analysis.
- This presentation explores:
  - Limit definition
  - Series expansion
  - Continued fractions
  - Numerical methods (e.g., Newton's method)
  - Power Ratio Method (PRM)
- Methods are analyzed for accuracy and convergence.

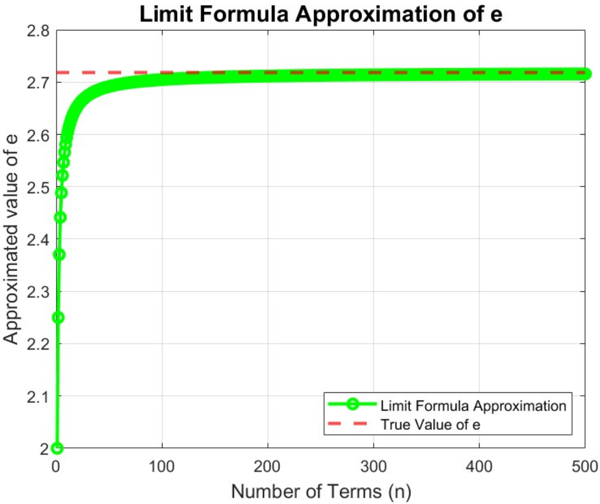
Limit Definition of e

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

- Originally derived from compound interest.
- Evaluated for increasing values of n:

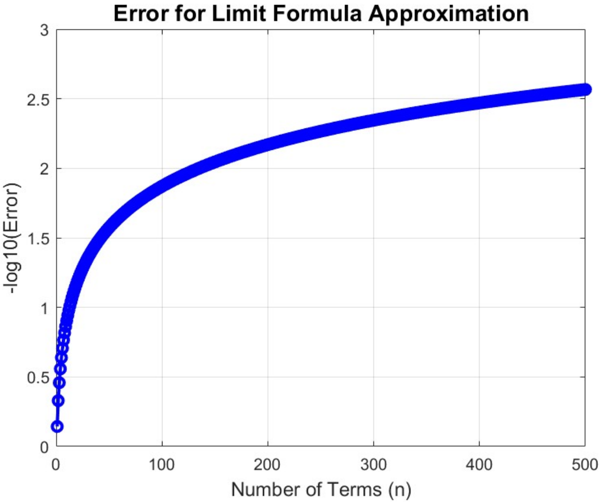
$$\begin{aligned} \text{For } n = 10, &\quad 2.59374 \\ \text{For } n = 100, &\quad 2.70481 \\ \text{For } n = 1000, &\quad 2.71692 \\ \text{As } n \to \infty, &\quad 2.71828 \end{aligned}$$

Graph of Limit Definition of e



Graph shows us convergence at around past 100 terms.

Graph of Error on the Limit Definition of e



Graph shows us this method is accurate to about 2.5 decimal places.

Maclaurin Series for e

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for } x = 1:$$

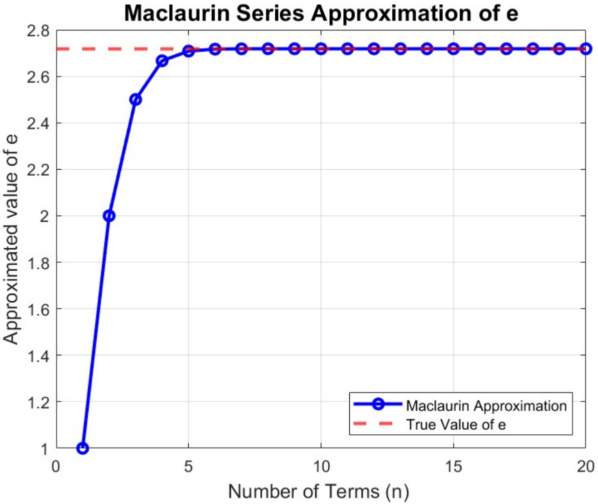
$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

- Approximation using terms:

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

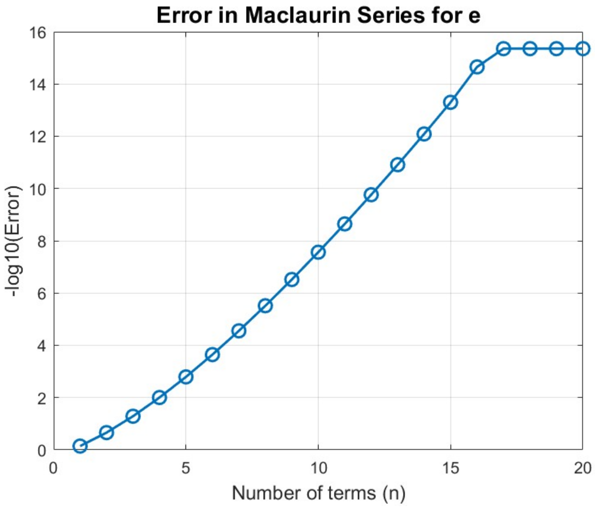
- Converges faster than the limit definition.

Graph of Maclaurin Series Approximation



Graph shows us the this method converges at about term 6.

Graph of Error on the Maclaurin Series Approximation



Graph shows us the method is accurate to about 15.3 decimal places.

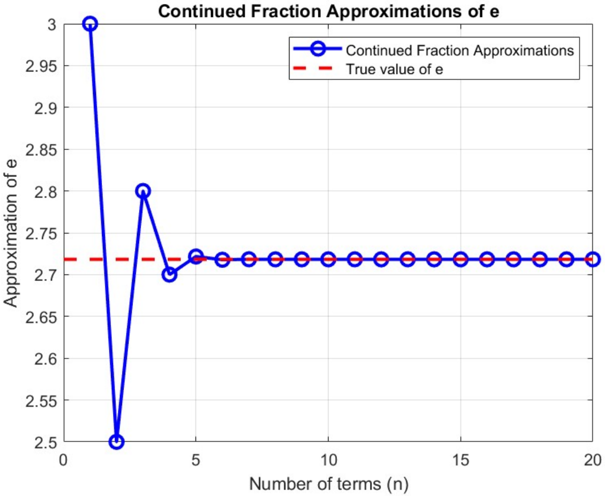
Continued Fraction for e

Euler developed the continued fraction representation of e:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

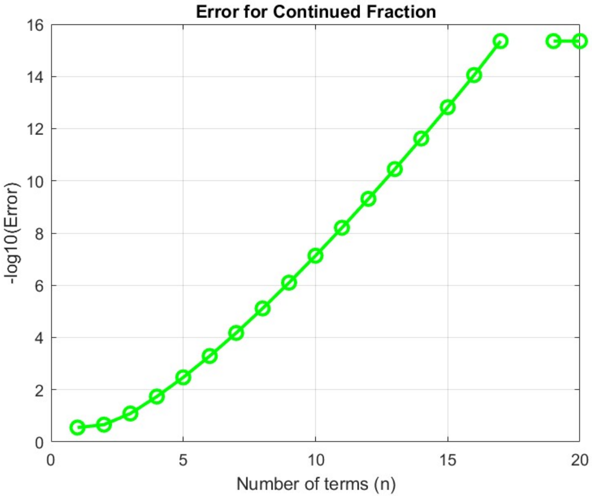
- Offers a compact and elegant approximation.
- Converges efficiently with increasing terms.

Graph of Continued Fractions Approximation



Graph shows us the method is rapidly converging in about 5 terms.

Graph of Error on the Continued Fractions Approximation



Graph shows us the method is accurate to about 15.3 decimal places.

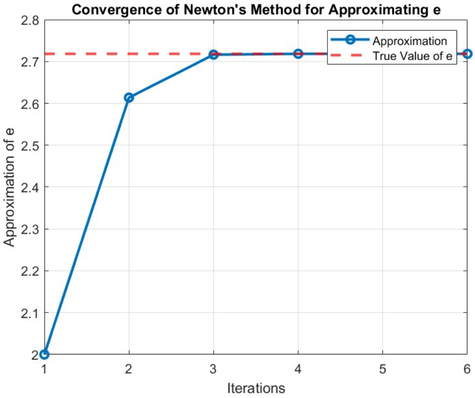
Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{where } f(x) = ln(x) - 1 = 0 \tag{1}$$

- Uses iterative refinement for fast convergence.
- Example:

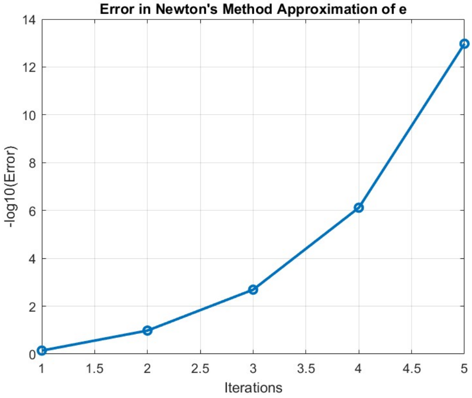
$$x_0 = 2.0000, \quad x_1 = 2.6137, \quad x_2 = 2.7162$$

Graph of Newton's Method for the Approximation of e



Graph showing rapid convergence of Newton's method for approximating e in about 3 iterations.

Graph of Error for Newton's Method for the Approximation of e



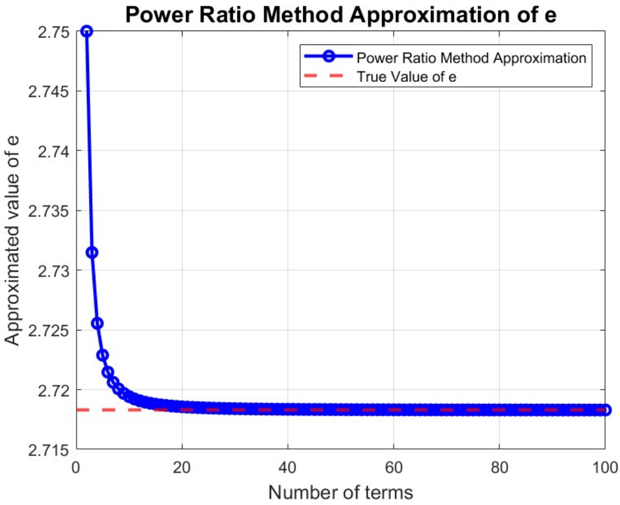
This graph shows us after 5 iterations that this method is accurate to about 12.9 decimal places.

Power Ratio Method

$$e \approx \frac{(x+1)^{x+1}}{x^x} - \frac{x^x}{(x-1)^{x-1}}$$

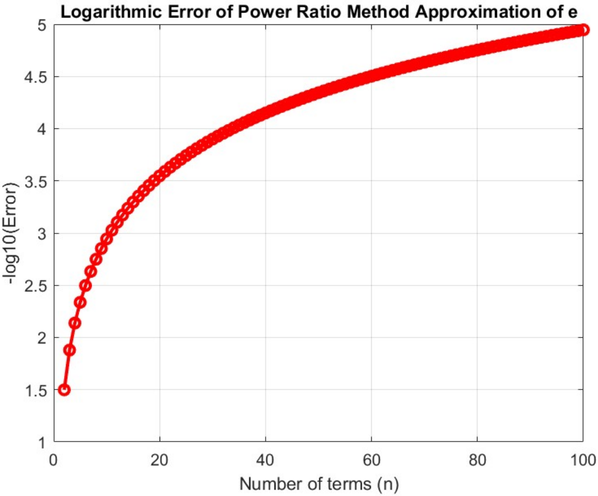
- This method was discovered by investigating the behavior of numbers raised to their own power. When we examine the rate of change of the ratio between adjacent integer values of x that have been raised to the x power lead to the approximation of e.[2]

Graph of Power Ratio Method for Approximating e



Graph comparing convergence of Power Ratio Method for approximating e.

Graph of Error in Power Ratio Method for Approximating *e*

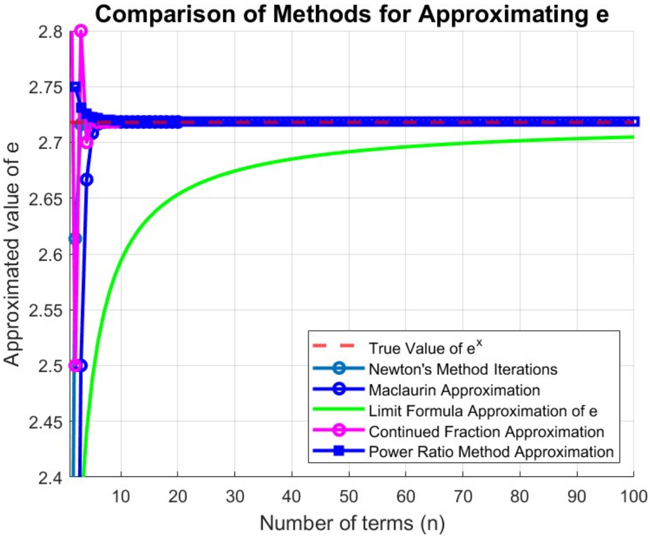


Graph shows that this method is accurate to about 5 decimal places.

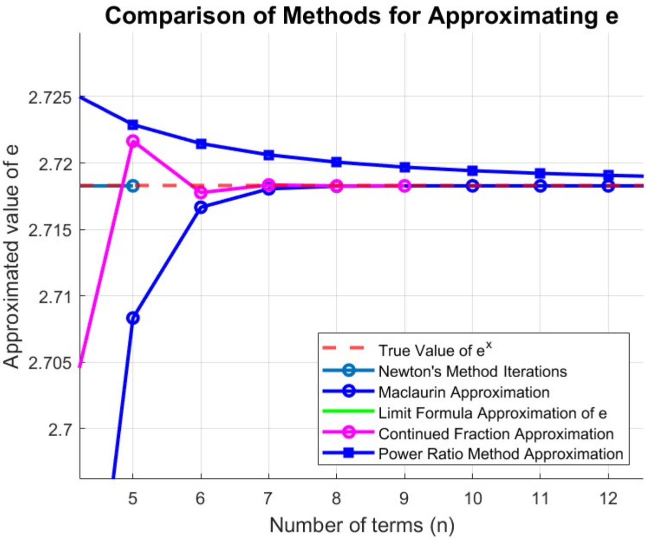
Comparison of Methods

- **Limit Definition:** Intuitive, but slow convergence.
- **Maclaurin Series:** Faster than the limit, good for computation.
- **Continued Fraction:** Compact, efficient.
- **Newton's Method:** Rapid convergence.
- **Power Ratio Method:** Unique perspective, good convergence.

Graph of Comparisons for Approximating e



Graph of Comparisons for Approximating e Zoomed In



Conclusion

- Each method has unique strengths, weaknesses, and applications.
- Newton's Method, the Power Ratio Method, Continued Fractions approximation, and the Maclaurin approximation converge rapidly.
- Newton's Method, Power Ratio Method, and the Limit formula for approximation are not as accurate compared to the other methods .

References

McCartin, B.J. ”e: The Master of All.” *The Mathematical Intelligencer*, 2006.

Brothers, H.J., and Knox, J.A. ”New Closed-Form Approximations.” *The Mathematical Intelligencer*, 1998.